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Decoherence of quantum information in the non-Markovian qubit channel

Masashi Ban[1,2], **Sachiko Kitajima**[3] **and Fumiaki Shibata**[3]

[1] Advanced Research Laboratory, Hitachi Ltd, 2520 Akanuma, Hatoyama, Saitama 350-0395, Japan
[2] CREST, Japan Science and Technology Agency, 1-1-9 Yaesu, Chuo-ku, Tokyo 103-0028, Japan
[3] Department of Physics, Faculty of Science, Ochanomizu University, 2-1-1 Ohtsuka, Bunkyo-ku, Tokyo 112-8610, Japan

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Abstract

Decoherence of quantum information of qubits is investigated under the influence of the non-Markovian quantum channel, where the correlation time of reservoir variables takes a finite value. Degradation of purity, distinguishability and entanglement of qubit states are evaluated. It is found that the non-Markov effect makes the coherence time of quantum information longer than that obtained for the Markovian quantum channel. Furthermore, the quantum teleportation and quantum dense coding of qubits are considered under the influence of non-Markov channels. The fidelity between teleported and original states and the Holevo capacity are obtained.

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1. Introduction

There have been considerable advances in the field of quantum information sciences [1–3]. Among others, quantum information processing is essential in quantum cryptography, quantum communication and quantum computing. These are important not only in their useful aspect of applications but also in their relevance to basic principles of quantum mechanics. However, quantum mechanical systems are quite delicate in the sense that noises due to their environmental fluctuations give rise to loss of information called decoherence. That is, the quantum systems inevitably suffer losses of purity, indistinguishable property and degree of entanglement from the quantum and thermal noises due to the environment. Effects of noises are treated by several methods ranging from phenomenological description to microscopic theory. In the phenomenological treatments [4–9], the loss of coherence is characterized by several energy and phase relaxation times, for instance, they are the longitudinal and the transverse relaxation times of the Bloch equations. These phenomenological methods are extended to treat the dynamical effects of environment in terms of stochastic processes, typical examples of which are the Gauss–Markov and the two-state jump Markov (random telegram) processes. In the microscopic level of description, the environment itself is considered to be a quantum system with quantum mechanical reservoir variables which are to be eliminated, for examples, by the method of projection operators [10–12] or by the path integral method [13]. Although in many cases, the Markovian approximation is assumed when the projection operator method is applied; it has recently been shown that the non-Markovian effect plays an important role in the relaxation processes [14–19].

The present authors have recently considered the non-equilibrium dynamics of quantum information by means of the phenomenological method [9] and the stochastic method [20]. The Bloch channel and the Kubo–Anderson channel of qubits have been introduced and their properties have been investigated in detail [9, 20]. In this paper, we will investigate decoherence of quantum information under the influence of a thermal reservoir, using the microscopic approach which assumes a system–reservoir interaction and applied the projector operator method to eliminate the reservoir variables. In particular, we pay our attention to the non-Markovian effect of the thermal reservoir, where the correlation time of reservoir variables is assumed to take a finite value. This paper is organized as follows. In section 2, we derive a non-Markovian quantum channel of qubits that becomes the well-known Markovian quantum channel in the limit that the correlation time reduces to zero. In sections 3 and 4, we investigate the degradation of purity, distinguishability and entanglement of qubit states under the influence of the non-Markovian quantum channel. The results show that the non-Markovian effect suppresses their degradation. In section 5, we consider the quantum teleportation of qubits and calculate the fidelity to show how faithfully a qubit state is teleported. In section 6, we investigate the transmission of classical information in the quantum dense coding of qubits. We obtain the Holevo capacity and compare it with the Shannon mutual information by the Bell measurement. We give concluding remarks in section 7.

2. Non-Markovian quantum channel of qubits

This section derives the non-Markovian quantum channel of qubits by means of the projection operator method [10–12]. A quantum state $\hat{W}(t)$ of a single qubit and reservoir–system is subject to the Liouville–von Neumann equation

$$\frac{\partial}{\partial t}\hat{W}(t) = -\frac{\mathrm{i}}{\hbar}[\hat{H}_Q + \hat{H}_{QR} + \hat{H}_R, \hat{W}(t)] \tag{1}$$

where $\hat{H}_Q$ and $\hat{H}_R$ are the Hamiltonians of a single qubit and reservoir and $\hat{H}_{QR}$ is the Hamiltonian between them. We assume that the interaction Hamiltonian $\hat{H}_{QR}$ is given by

$$\hat{H}_{QR} = \hbar\lambda(\hat{\sigma}_+\hat{R} + \hat{\sigma}_-\hat{R}^\dagger) \tag{2}$$

where $\hat{\sigma}_\pm = \hat{\sigma}_x \pm \mathrm{i}\hat{\sigma}_y$ with $\hat{\sigma}_k$ $(k = x, y, z)$ being the Pauli matrix and $\hat{R}$ stands for some reservoir operator. When we use the projection operator method to eliminate the reservoir operators from equation (1), we can obtain the time-convolutionless quantum master equation of the qubit up to the second order with respect to the coupling constant λ in the interaction picture [10–12]

$$\begin{aligned}\frac{\partial}{\partial t}\hat{\rho}(t) = \phi^*_{+-}(t)[\hat{\sigma}_+, \hat{\rho}(t)\hat{\sigma}_-] + \phi_{+-}(t)[\hat{\sigma}_+\hat{\rho}(t), \hat{\sigma}_-] \\ + \phi^*_{-+}(t)[\hat{\sigma}_-, \hat{\rho}(t)\hat{\sigma}_+] + \phi_{-+}(t)[\hat{\sigma}_-\hat{\rho}(t), \hat{\sigma}_+]\end{aligned} \tag{3}$$

where $\hat{\rho}(t) = \mathrm{Tr}_R\,\hat{W}(t)$ is the reduced density operator of the single qubit. The functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$ are given by

$$\phi_{+-}(t) = \lambda^2\int_0^t \mathrm{d}\tau\,\mathrm{e}^{-\mathrm{i}\omega_Q\tau}\langle\hat{R}^\dagger(\tau)\hat{R}(0)\rangle_R \tag{4}$$

$$\phi_{-+}(t) = \lambda^2 \int_0^t d\tau\, e^{i\omega_Q \tau} \langle \hat{R}(\tau)\hat{R}^\dagger(0)\rangle_R \tag{5}$$

where $\hbar\omega_Q$ is the energy difference between the two levels of the qubit and $\langle\cdots\rangle_R$ stands for the average value in the thermal equilibrium of the reservoir. In the Markovian limit, the functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$ are replaced with $\phi_{+-}(\infty)$ and $\phi_{-+}(\infty)$ in equation (3).

To determine the quantum master equation (3), we must find the functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$ given by equations (4) and (5). This is equivalent to fixing the model of the thermal reservoir. In this paper, we assume that the correlation functions of the reservoir variables decay exponentially with the relaxation time τ_R, that is

$$\lambda^2 \langle \hat{R}^\dagger(t)\hat{R}(0)\rangle_R = \frac{1}{\tau_R} G_{+-}\, e^{i\omega_R t - t/\tau_R} \tag{6}$$

$$\lambda^2 \langle \hat{R}(t)\hat{R}^\dagger(0)\rangle_R = \frac{1}{\tau_R} G_{-+}\, e^{-i\omega_R t - t/\tau_R}. \tag{7}$$

Substituting these equations into equations (4) and (5), we can obtain the functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$

$$\phi_{+-}(t) = G_{+-} \frac{1 - \exp(-(1 + i(\omega_Q - \omega_R)\tau_R)t/\tau_R)}{1 + i(\omega_Q - \omega_R)\tau_R} \tag{8}$$

$$\phi_{-+}(t) = G_{-+} \frac{1 - \exp(-(1 - i(\omega_Q - \omega_R)\tau_R)t/\tau_R)}{1 - i(\omega_Q - \omega_R)\tau_R} \tag{9}$$

which are formally rewritten as

$$\phi_{+-}(t) = \phi_{+-}(\infty)(1 - \exp(-(1 + i(\omega_Q - \omega_R)\tau_R)t/\tau_R)) \tag{10}$$

$$\phi_{-+}(t) = \phi_{+-}(\infty)(1 - \exp(-(1 - i(\omega_Q - \omega_R)\tau_R)t/\tau_R)). \tag{11}$$

Furthermore we assume the resonant $\omega_Q = \omega_R$ or nearly resonant such that $|\omega_Q - \omega_R|\tau_R \ll 1$. Hence we finally obtain the functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$

$$\phi_{+-}(t) = \phi_{+-}(\infty)\tau_R f'(t) \tag{12}$$

$$\phi_{-+}(t) = \phi_{-+}(\infty)\tau_R f'(t) \tag{13}$$

where $f'(t) = df(t)/dt$ and the function $f(t)$ is given by

$$f(t) = \frac{t}{\tau_R} - 1 + e^{-t/\tau_R} \tag{14}$$

which satisfies $\lim_{\tau_R \to 0} \tau_R f'(t) = 1$ and $\lim_{\tau_R \to 0} \tau_R f(t) = t$. This result is equivalent to that obtained when the thermal reservoir is a set of damped oscillators in the thermal equilibrium, which will be discussed in appendix.

Substituting equations (12) and (13) into equation (3) and neglecting the small frequency shift caused by the thermal reservoir, we obtain the non-Markovian quantum master equation of a single qubit

$$\frac{\partial}{\partial t}\hat{\rho}(t) = \frac{1 + \langle\hat{\sigma}_z\rangle_{eq}}{2T_2} \tau_R f'(t)([\hat{\sigma}_+, \hat{\rho}(t)\hat{\sigma}_-] + [\hat{\sigma}_+\hat{\rho}(t), \hat{\sigma}_-]) + \frac{1 - \langle\hat{\sigma}_z\rangle_{eq}}{2T_2} \tau_R f'(t)([\hat{\sigma}_-, \hat{\rho}(t)\hat{\sigma}_+] + [\hat{\sigma}_-\hat{\rho}(t), \hat{\sigma}_+]) \tag{15}$$

with

$$T_2^{-1} = \mathrm{Re}[\phi_{+-}(\infty) + \phi_{-+}(\infty)] \tag{16}$$

$$\langle\hat{\sigma}_z\rangle_{eq} = -\tanh\left(\frac{\hbar\omega_Q}{2k_B T}\right). \tag{17}$$

In the limit of $\tau_R \to 0$, equation (15) reduces to the well-known Markovian quantum master equation of a single qubit with the transverse relaxation time T_2 and the longitudinal relaxation time $T_1 = T_2/2$.

To obtain the non-Markovian quantum channel of a qubit, we first calculate the average values $\langle\hat{\sigma}_\pm\rangle_t = \mathrm{Tr}[\hat{\sigma}_\pm\hat{\rho}(t)]$ and $\langle\hat{\sigma}_z\rangle_t = \mathrm{Tr}[\hat{\sigma}_z\hat{\rho}(t)]$ which are subject to the Bloch-like equations from equation (15)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_\pm\rangle_t = -\frac{\tau_R f'(t)}{T_2}\langle\hat{\sigma}_\pm\rangle_t \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_z\rangle_t = -\frac{2\tau_R f'(t)}{T_2}[\langle\hat{\sigma}_z\rangle_t - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}] \tag{19}$$

where we have ignored the angular frequency ω_Q of the qubit since it does not affect purity and entanglement of the qubit states. Then we find the solutions

$$\langle\hat{\sigma}_\pm\rangle_t = \gamma_t\langle\hat{\sigma}_\pm\rangle_0 \tag{20}$$

$$\langle\hat{\sigma}_z\rangle_t = \gamma_t^2\langle\hat{\sigma}_z\rangle_0 + \left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} \tag{21}$$

where the parameter γ_t is given by

$$\gamma_t = \mathrm{e}^{-(\tau_R/T_2)f(t)}. \tag{22}$$

Any single qubit state $\hat{\rho}(t)$ at time t can be expressed in terms of $\langle\hat{\sigma}_\pm\rangle_t$ and $\langle\hat{\sigma}_z\rangle_t$ as

$$\hat{\rho}(t) = \tfrac{1}{2}(\hat{1} + \langle\hat{\sigma}_-\rangle_t\hat{\sigma}_+ + \langle\hat{\sigma}_+\rangle_t\hat{\sigma}_- + \langle\hat{\sigma}_z\rangle_t\hat{\sigma}_z). \tag{23}$$

Hence the non-Markovian qubit channel $\hat{\mathcal{L}}_t$ defined by the relation $\hat{\rho}(t) = \hat{\mathcal{L}}_t\hat{\rho}(0)$ for any initial state $\hat{\rho}(0)$ is determined by the relations

$$\hat{\mathcal{L}}_t|0\rangle\langle 0| = \tfrac{1}{2}\left(1+\gamma_t^2\right)|0\rangle\langle 0| + \tfrac{1}{2}\left(1-\gamma_t^2\right)|1\rangle\langle 1| + \tfrac{1}{2}\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}(|0\rangle\langle 0| - |1\rangle\langle 1|) \tag{24}$$

$$\hat{\mathcal{L}}_t|1\rangle\langle 1| = \tfrac{1}{2}\left(1-\gamma_t^2\right)|0\rangle\langle 0| + \tfrac{1}{2}\left(1+\gamma_t^2\right)|1\rangle\langle 1| + \tfrac{1}{2}\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}(|0\rangle\langle 0| - |1\rangle\langle 1|) \tag{25}$$

$$\hat{\mathcal{L}}_t|0\rangle\langle 1| = \gamma_t|0\rangle\langle 1| \tag{26}$$

$$\hat{\mathcal{L}}_t|1\rangle\langle 0| = \gamma_t|1\rangle\langle 0| \tag{27}$$

with $\hat{\sigma}_z|0\rangle = |0\rangle$ and $\hat{\sigma}_z|1\rangle = -|1\rangle$. We also obtain the Kraus form of the non-Markovian quantum channel $\hat{\mathcal{L}}_t$

$$\hat{\mathcal{L}}_t\hat{X} = p_0(t)\hat{X} + \sum_{k=x,y,z} p_k(t)\hat{\sigma}_k\hat{X}\hat{\sigma}_k \tag{28}$$

where $p_0(t)$, $p_x(t)$, $p_y(t)$ and $p_z(t)$ are given by

$$p_0(t) = \tfrac{1}{4}(1+\gamma_t)^2 + \tfrac{1}{4}\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} \tag{29}$$

$$p_x(t) = \tfrac{1}{4}\left(1-\gamma_t^2\right)[1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}] \tag{30}$$

$$p_y(t) = \tfrac{1}{4}\left(1-\gamma_t^2\right)[1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}] \tag{31}$$

$$p_z(t) = \tfrac{1}{4}(1-\gamma_t)^2 + \tfrac{1}{4}\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}. \tag{32}$$

In the limit of $\tau_R \to 0$, the parameter γ_t is replaced with e^{-t/T_2} and the quantum channel $\hat{\mathcal{L}}_t$ becomes the well-known Markovian quantum channel [9].

3. Decay of purity and distinguishability of qubit states

This section investigates degradation of purity and distinguishability of qubit states under the influence of the non-Markovian quantum channel $\hat{\mathcal{L}}_t$ given by equations (24)–(27). Purity of a qubit state $\hat{\rho}(t)$ is evaluated by the linear entropy $S_L(\hat{\rho}(t)) = 1 - \mathrm{Tr}\,\hat{\rho}^2(t)$ which is expressed in terms of the Bloch vector $\vec{a}(t)$ of the quantum state $\hat{\rho}(t)$ as

$$S_L(\hat{\rho}(t)) = \tfrac{1}{2}(1 - |\vec{a}(t)|^2). \tag{33}$$

Since $\hat{\rho}(t) = \hat{\mathcal{L}}_t\hat{\rho}(0)$, the Bloch vector $\vec{a}(t)$ is related to $\vec{a}(0)$ by the relation

$$\vec{a}(t) = \mathsf{L}_t\vec{a}(0) + \vec{b}_t \tag{34}$$

with

$$\mathsf{L}_t = \begin{pmatrix} \gamma_t & 0 & 0 \\ 0 & \gamma_t & 0 \\ 0 & 0 & \gamma_t^2 \end{pmatrix} \qquad \vec{b}_t = \begin{pmatrix} 0 \\ 0 \\ \left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} \end{pmatrix}. \tag{35}$$

If the initial state $\hat{\rho}(0)$ is pure and thus $|\vec{a}(0)|^2 = 1$ is satisfied, we obtain

$$|\vec{a}(t)|^2 = \gamma_t^2 - \gamma_t^2\left(1-\gamma_t^2\right)a_z^2(0) + 2\gamma_t^2\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}a_z(0) + \left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2. \tag{36}$$

When we take the average over all possible pure qubit states, we obtain $\overline{a_k(0)} = 0$ and $\overline{a_k^2(0)} = 1/3$ ($k = x, y, z$). Then the averaged linear entropy $\overline{S_L(\hat{\rho}(0))}$ for initial pure states is given by

$$\overline{S_L(\hat{\rho}(0))} = \tfrac{1}{2}\left(1-\gamma_t^2\right)\left[1 + \tfrac{1}{3}\gamma_t^2 - \left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2\right]. \tag{37}$$

We find from this equation that when $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2 \leqslant 1/3$, the averaged linear entropy decreases monotonously with time t while when $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2 > 1/3$, it takes the maximum value at the time t_m which is determined by the relation

$$\gamma_{t_m}^2 = \frac{\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2 - 1/3}{\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2 + 1/3}. \tag{38}$$

The averaged linear entropy $\overline{S_L(\hat{\rho}(t))}$ is plotted as the function of time t in figure 1. The figure clearly shows that the non-Markovian effect suppresses the degradation of purity of qubit states. In the case of $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = -1$, we have $\overline{S_L(\hat{\rho}(\infty))} = 0$ since $\hat{\rho}(\infty) = |1\rangle\langle 1|$.

Distinguishability or similarity between two quantum states $\hat{\rho}_1$ and $\hat{\rho}_2$ can be measured by means of the fidelity

$$F(\hat{\rho}_1, \hat{\rho}_2) = \left[\mathrm{Tr}(\sqrt{\hat{\rho}_1}\hat{\rho}_2\sqrt{\hat{\rho}_1})^{1/2}\right]^2. \tag{39}$$

For two qubit states which have the Bloch vectors $\vec{a}_1$ and $\vec{a}_2$, the fidelity can be expressed as

$$F(\hat{\rho}_1, \hat{\rho}_2) = \tfrac{1}{2}(1 + \vec{a}_1\cdot\vec{a}_2) + \tfrac{1}{2}\sqrt{(1-|\vec{a}_1|^2)(1-|\vec{a}_2|^2)}. \tag{40}$$

Another measure of distinguishability between two quantum states is the average probability of error in the optimum quantum measurement. The quantum detection theory provides [21]

$$P_{\mathrm{error}} = \tfrac{1}{2} - \tfrac{1}{4}\|\hat{\rho}_1 - \hat{\rho}_2\|_1 \tag{41}$$

where $\|\hat{X}\|_1 = \mathrm{Tr}\sqrt{\hat{X}^\dagger\hat{X}}$. For qubit states, we have

$$P_{\mathrm{error}} = \tfrac{1}{2} - \tfrac{1}{4}|\vec{a}_1 - \vec{a}_2|. \tag{42}$$

Since any two quantum states that satisfy $\hat{\rho}_1\hat{\rho}_2 = 0$ can be discriminated without error, $F(\hat{\rho}_1, \hat{\rho}_2) = 0$ and $P_{\mathrm{error}} = 0$ are obtained.

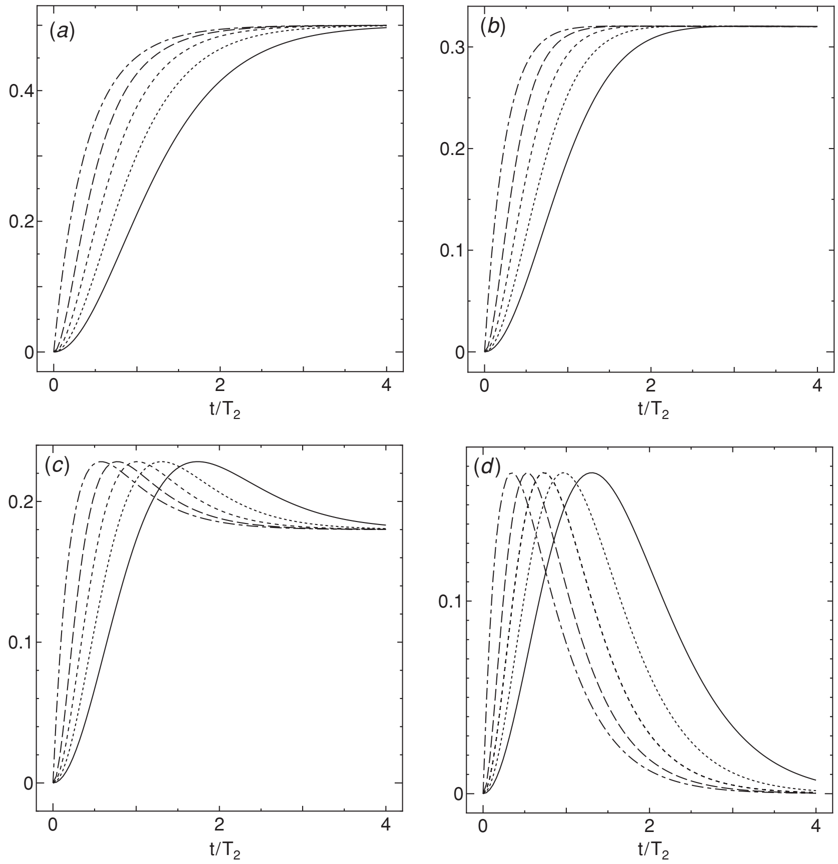

Figure 1. Time-evolution of the averaged linear entropy $\overline{S_L(\hat{\rho}(t))}$ for initial pure states, where (*a*) $\langle\hat{\sigma}_z\rangle_{\rm eq} = 0.0$, (*b*) $\langle\hat{\sigma}_z\rangle_{\rm eq} = -0.6$, (*c*) $\langle\hat{\sigma}_z\rangle_{\rm eq} = -0.8$ and (*d*) $\langle\hat{\sigma}_z\rangle_{\rm eq} = -1.0$. In each figure, the solid line represents $\tau_R/T_2 = 2.0$, the dotted line $\tau_R/T_2 = 1.0$, the short dashed line $\tau_R/T_2 = 0.5$, the dashed line $\tau_R/T_2 = 0.2$, and the dot-dashed line $\tau_R/T_2 = 0.0$ which corresponds to the Markovian approximation.

We suppose that two qubit states at the initial time are $|\psi_1\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|\psi_2\rangle = |0\rangle$, where $\hat{\sigma}_x|\psi_1\rangle = |\psi_1\rangle$ and $\hat{\sigma}_z|\psi_2\rangle = |\psi_2\rangle$. When $\langle\hat{\sigma}_z\rangle_{\rm eq} = 1/2$, the Bloch vectors of the quantum states $\hat{\rho}_k = \hat{\mathcal{L}}_t|\psi_k\rangle\langle\psi_k|$ $(k = 1, 2)$ are given by

$$\vec{a}_1 = \begin{pmatrix} \gamma_t \\ 0 \\ -\frac{1}{2}(1-\gamma_2) \end{pmatrix} \qquad \vec{a}_2 = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}\left(1+\gamma_t^2\right) \end{pmatrix}. \tag{43}$$

Hence the fidelity $F(\hat{\rho}_1, \hat{\rho}_2)$ and the average probability of error $P_{\rm error}$ becomes

$$F(\hat{\rho}_1, \hat{\rho}_2) = 1 - \tfrac{1}{4}\gamma_t^2\left(1+\gamma_t^2\right) \tag{44}$$

$$P_{\rm error} = \tfrac{1}{2} - \tfrac{1}{4}\gamma_t\sqrt{1+\gamma_t^2} \tag{45}$$

which are plotted as the functions of time t in figure 2. It is seen from the figure that the non-Markovian effect also suppresses the degradation of the distinguishability between quantum states.

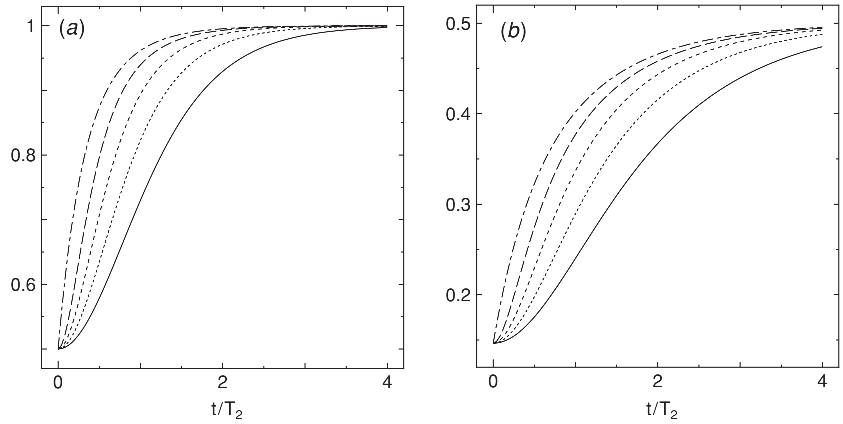

Figure 2. Time-evolution of (*a*) the fidelity and (*b*) the average probability of error, where $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = -1/2$. In each figure, the solid line represents $\tau_R/T_2 = 2.0$, the dotted line $\tau_R/T_2 = 1.0$, the short dashed line $\tau_R/T_2 = 0.5$, the dashed line $\tau_R/T_2 = 0.2$, and the dot-dashed line $\tau_R/T_2 = 0.0$ which corresponds to the Markovian approximation.

4. Decoherence of entanglement of qubits

In this section, we investigate the influence of the non-Markovian effect on the decoherence of entanglement of the Bell states. For this purpose, we suppose that one of the two qubits in the Bell states is sent through the non-Markovian quantum channel $\hat{\mathcal{L}}_t$ during time t. As the result, we obtain the mixed Bell states

$$\begin{aligned}\hat{\rho}_{\Phi_\pm}(t) = (\hat{\mathcal{L}}_t \otimes \hat{\mathcal{I}})|\Phi_\pm\rangle\langle\Phi_\pm| &= \tfrac{1}{4}(1 \pm \gamma_t)^2|\Phi_+\rangle\langle\Phi_+| + \tfrac{1}{4}(1 \mp \gamma_t)^2|\Phi_-\rangle\langle\Phi_-| \\ &+ \tfrac{1}{4}\left(1-\gamma_t^2\right)(|\Psi_+\rangle\langle\Psi_+| + |\Psi_-\rangle\langle\Psi_-|) \\ &+ \tfrac{1}{4}\left(1-\gamma_t^2\right)(|\Phi_+\rangle\langle\Phi_-| + |\Psi_+\rangle\langle\Psi_-| + \{\mathrm{h.c.}\})\end{aligned} \tag{46}$$

$$\begin{aligned}\hat{\rho}_{\Psi_\pm}(t) = (\hat{\mathcal{L}}_t \otimes \hat{\mathcal{I}})|\Psi_\pm\rangle\langle\Psi_\pm| &= \tfrac{1}{4}(1 \pm \gamma_t)^2|\Psi_+\rangle\langle\Psi_+| + \tfrac{1}{4}(1 \mp \gamma_t)^2|\Psi_-\rangle\langle\Psi_-| \\ &+ \tfrac{1}{4}\left(1-\gamma_t^2\right)(|\Phi_+\rangle\langle\Phi_+| + |\Phi_-\rangle\langle\Phi_-|) \\ &+ \tfrac{1}{4}\left(1-\gamma_t^2\right)(|\Phi_+\rangle\langle\Phi_-| + |\Psi_+\rangle\langle\Psi_-| + \{\mathrm{h.c.}\})\end{aligned} \tag{47}$$

with $|\Phi_\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Psi_\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. In these equations, {h.c.} stands for the Hermitian conjugate of $|\Phi_+\rangle\langle\Phi_-| + |\Psi_+\rangle\langle\Psi_-|$. The concurrence C of a bipartite state $\hat{\rho}$ is given in terms of the eigenvalues λ_k $(k = 1, 2, 3, 4)$ of $\hat{R} = (\sqrt{\hat{\rho}}\hat{\rho}'\sqrt{\hat{\rho}})^{1/2}$ with $\hat{\rho}' = (\hat{\sigma}_y \otimes \hat{\sigma}_y)\hat{\rho}^*(\hat{\sigma}_y \otimes \hat{\sigma}_y)$ [22, 23]

$$C = \max\left[0, 2\max_{1\leqslant k\leqslant 4}\lambda_k - \sum_{k=1}^{4}\lambda_k\right]. \tag{48}$$

Note that λ_k $(k = 1, 2, 3, 4)$ is equal to the square root of the eigenvalue of $\hat{\rho}\hat{\rho}'$ [23]. The eigenvalues of the matrix $\hat{R}$ with $\hat{\rho}_{\Phi_\pm}(t)$ and $\hat{\rho}_{\Psi_\pm}(t)$ are calculated to be

$$\lambda_{1,2} = \tfrac{1}{4}\left[\sqrt{(1+\gamma_t)^2 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2\left(1-\gamma_t^2\right)} \pm 2\gamma_t\right] \tag{49}$$

$$\lambda_{3,4} = \tfrac{1}{4}\sqrt{1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2\left(1-\gamma_t^2\right)}. \tag{50}$$

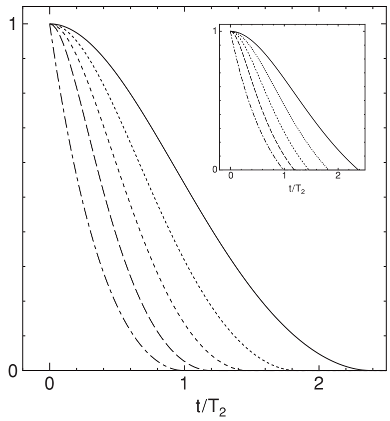

Figure 3. Decay of the entanglement of formation under the influence of the non-Markovian quantum channel, where the entanglement is measured in *ebits* and $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = -1/2$. In the figure, the solid line represents $\tau_R/T_2 = 2.0$, the dotted line $\tau_R/T_2 = 1.0$, the short dashed line $\tau_R/T_2 = 0.5$, the dashed line $\tau_R/T_2 = 0.2$, and the dot-dashed line $\tau_R/T_2 = 0.0$ which corresponds to the Markovian approximation. The inset graph shows the concurrence in the same parameters.

Using the results, we can obtain the concurrence C_t of the mixed Bell states given by equations (46) and (47)

$$C_t = \max\left[0, \gamma_t - \tfrac{1}{2}\sqrt{1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2}\left(1 - \gamma_t^2\right)\right]. \tag{51}$$

Then the entanglement of formation E_t [22, 23, 26] of the mixed Bells states is given by

$$E_t = H\left(\frac{1 + \sqrt{1 - C_t^2}}{2}\right) \tag{52}$$

where $H(x) = -x\log_2 - (1-x)\log_2(1-x)$. The entanglement of formation is plotted as the function of time t in figure 3. The figure shows that the entanglement decays most rapidly in the Markovian approximation. As the correlation time τ_R of the reservoir variables is greater, the entanglement survives longer.

It is found from equations (22) and (51) that the entanglement of the quantum states $\hat{\rho}_{\Phi_\pm}(t)$ and $\hat{\rho}_{\Psi_\pm}(t)$ is completely destructed at the time t_e which is determined by the relation

$$\frac{t_e}{T_2} - \frac{\tau_R}{T_2}\left(1 - \mathrm{e}^{-t_e/\tau_R}\right) = \ln\left[\frac{\sqrt{2 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2} + 1}{\sqrt{1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2}}\right]. \tag{53}$$

In the Markovian limit ($\tau_R \to 0$), this equation reduces to

$$\frac{t_e'}{T_2} = \ln\left[\frac{\sqrt{2 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2} + 1}{\sqrt{1 - \langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2}}\right]. \tag{54}$$

This result implies that when $t \geqslant t_e$, the non-Markovian quantum channel $\hat{\mathcal{L}}_t$ becomes an entanglement-breaking channel [24, 25]. Furthermore we find that $t_e \to \infty$ at $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2 \to 1$.

This indicates that nonzero temperature of the thermal reservoir is indispensable for the non-Markovian quantum channel to be entanglement-breaking. From equations (53) and (54), we obtain the relation

$$t_e - \tau_R(1 - \mathrm{e}^{-t_e/\tau_R}) = t'_e \tag{55}$$

which provides the inequality $t_e > t'_e$. The non-Markovian effect makes longer the time after which the non-Markovian quantum channel $\hat{\mathcal{L}}_t$ becomes entanglement-breaking. The inequality $t_e > t'_e$ together with equation (54) yields

$$t_e > t'_e > T_2. \tag{56}$$

When the transverse relaxation time T_2 of the qubit is large in comparison with the correlation time τ_R of the reservoir variables, we can obtain the approximated relation $t_e - t'_e \approx \tau_R$. In this case, the time at which the entanglement disappears is made larger by τ_R due to the non-Markovian effect.

5. Fidelity of qubit states in quantum teleportation

We now consider the quantum teleportation of qubit states [27] under the influence of the non-Markovian quantum channel. We suppose that a sender, Alice, sends to a receiver, Bob, one of the two qubits in the Bell state $|\Phi_+\rangle$ through the non-Markovian quantum channel $\hat{\mathcal{L}}_t$ to share the entanglement with him, where t represents the transmission time of the qubit. In this case, Alice and Bob share the mixed Bell state $\hat{\rho}_{\Phi_+}(t)$ given by equation (46). When Alice teleports an arbitrary qubit state $\hat{\rho}_{\rm in}$ to Bob, where the standard protocol is applied, he can obtain the quantum state $\hat{\rho}_{\rm out}$ in average [28]

$$\hat{\rho}_{\rm out} = P_0\hat{\rho}_{\rm in} + P_x\hat{\sigma}_x\hat{\rho}_{\rm in}\hat{\sigma}_x + P_y\hat{\sigma}_y\hat{\rho}_{\rm in}\hat{\sigma}_y + P_z\hat{\sigma}_z\hat{\rho}_{\rm in}\hat{\sigma}_z \tag{57}$$

where P_0, P_x, P_y and P_z are given by

$$P_0 = \langle\Phi_+|\hat{\rho}_{\Phi_+}|\Phi_+\rangle = \tfrac{1}{4}(1+\gamma_t)^2 \tag{58}$$

$$P_x = \langle\Psi_+|\hat{\rho}_{\Phi_+}|\Psi_+\rangle = \tfrac{1}{4}\left(1-\gamma_t^2\right) \tag{59}$$

$$P_y = \langle\Psi_-|\hat{\rho}_{\Phi_+}|\Psi_-\rangle = \tfrac{1}{4}\left(1-\gamma_t^2\right) \tag{60}$$

$$P_z = \langle\Phi_-|\hat{\rho}_{\Phi_+}|\Phi_-\rangle = \tfrac{1}{4}(1-\gamma_t)^2. \tag{61}$$

Since we can express any qubit state as $\hat{\rho}_{\rm in} = (1/2)(\hat{1} + a_x\hat{\sigma}_x + a_y\hat{\sigma}_y + a_z\hat{\sigma}_z)$, the quantum state that Bob can get in average becomes

$$\hat{\rho}_{\rm out} = \tfrac{1}{2}\left(\hat{1} + \gamma_t a_x\hat{\sigma}_x + \gamma_t a_y\hat{\sigma}_y + \gamma_t^2 a_x\hat{\sigma}_x\right). \tag{62}$$

Thus the quantum teleportation with the standard protocol is equivalent to the transformation of the Bloch vector $\vec{a} = (a_x, a_y, a_z)^{\rm T}$ of $\hat{\rho}_{\rm in}$

$$\vec{a} \longrightarrow \mathsf{L}_t\vec{a} \tag{63}$$

with

$$\mathsf{L}_t = \begin{pmatrix} \gamma_t & 0 & 0 \\ 0 & \gamma_t & 0 \\ 0 & 0 & \gamma_t^2 \end{pmatrix}. \tag{64}$$

We obtain the fidelity $F_t = [\mathrm{Tr}(\sqrt{\hat{\rho}_{\rm in}}\hat{\rho}_{\rm out}\sqrt{\hat{\rho}_{\rm in}})^{1/2}]^2$ to investigate how faithfully the quantum state $\hat{\rho}_{\rm in}$ is teleported from Alice to Bob. The fidelity F_t can be expressed in terms of the Bloch vector $\hat{a}$

$$F_t = \tfrac{1}{2}(1 + \vec{a}\cdot\mathsf{L}_t\vec{a}) + \tfrac{1}{2}\sqrt{(1-|\vec{a}|^2)(1-|\mathsf{L}_t\vec{a}|^2)}. \tag{65}$$

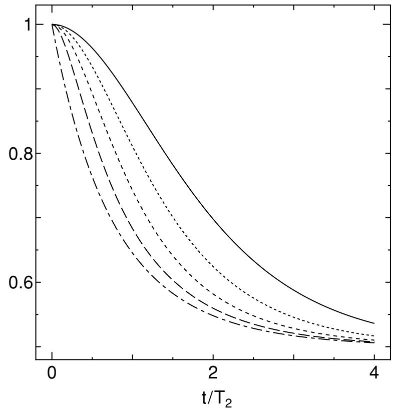

Figure 4. Dependence of the averaged fidelity $\bar{F}_t$ on the transmission time t. In the figure, the solid line represents $\tau_R/T_2 = 2.0$, the dotted line $\tau_R/T_2 = 1.0$, the short dashed line $\tau_R/T_2 = 0.5$, the dashed line $\tau_R/T_2 = 0.2$, and the dot-dashed line $\tau_R/T_2 = 0.0$ which corresponds to the Markovian approximation.

In particular, when the quantum state $\hat{\rho}_{\text{in}}$ is pure ($|\vec{a}| = 1$), the fidelity F_t becomes

$$F_t = \tfrac{1}{2}(1+\gamma_t) - \tfrac{1}{2}\gamma_t(1-\gamma_t)|a_z|^2. \tag{66}$$

When we take the average of the fidelity F_t over all possible pure qubit states, the averaged fidelity $\bar{F}_t$ is given by

$$\bar{F}_t = \tfrac{1}{6}\left(3 + 2\gamma_t + \gamma_t^2\right) \tag{67}$$

where we have used $\overline{|a_z|^2} = 1/3$. The averaged fidelity $\bar{F}_t$ is plotted as the function of the transmission time t in figure 4. In the classical teleportation, the upper bound on the averaged fidelity is $2/3$. Then the condition under which the quantum teleportation works well under the influence of the non-Markovian quantum channel is given by

$$\gamma_t > \sqrt{2} - 1. \tag{68}$$

It is easy to see from equations (51) and (52) that the condition is equivalent to the concurrence (or equivalently the entanglement of formation) of the quantum state $\hat{\rho}_{\Phi_+}(t)$ shared by Alice and Bob, which is greater than zero in the case of $\langle\hat{\sigma}_z\rangle_{\text{eq}} = 0$, that is, $C_{t|\langle\hat{\sigma}_z\rangle_{\text{eq}}=0} > 0$ and $E_{t|\langle\hat{\sigma}_z\rangle_{\text{eq}}=0} > 0$. It is important to note that $\langle\hat{\sigma}_z\rangle_{\text{eq}} = 0$ corresponds to the infinite temperature ($T \to \infty$) of the thermal reservoir (see equation (17)). By making use of the concurrence $C_{t|\langle\hat{\sigma}_z\rangle_{\text{eq}}=0}$, the averaged fidelity $\bar{F}_t$ can be expressed as

$$\bar{F}_t = \frac{2 + C_{t|\langle\hat{\sigma}_z\rangle_{\text{eq}}=0}}{3} \tag{69}$$

which implies that the entanglement always makes the averaged fidelity greater than that of the classical teleportation.

6. Capacity of quantum dense coding system

This section considers classical information transmission in the quantum dense coding system of qubits [29] under the influence of the non-Markovian quantum channel. Taking into account of the entanglement distillation [30], we may assume that Alice and Bob can share the Bell state $|\Phi_+\rangle$ even if there is a noisy environment. Hence we suppose that Alice and Bob share the Bell state $|\Phi_+\rangle$. Alice encodes two bits of classical information by applying the one of four operators $\hat{1}$, $\hat{\sigma}_z$, $\hat{\sigma}_x$ and $\hat{\sigma}_y$ to her qubit. Then Alice sends the encoded qubit to Bob through the non-Markovian quantum channel $\hat{\mathcal{L}}_t$, where t stands for the transmission time of the encoded qubit. After receiving it, Bob obtains one of the four two-qubit states $\hat{\rho}_{\Phi_\pm}(t)$ and $\hat{\rho}_{\Psi_\pm}(t)$ given by equations (46) and (47). Since the Holevo capacity [31, 32] is attained when the prior probabilities of two bits of classical information are equal [33], we obtain the Holevo capacity C_{H} of the quantum dense coding system

$$C_{\mathrm{H}} = S\left(\frac{1}{4}\sum_{k=\Phi_\pm,\Psi_\pm}\hat{\rho}_k(t)\right) - \frac{1}{4}\sum_{k=\Phi_\pm,\Psi_\pm} S(\hat{\rho}_k(t)) \tag{70}$$

where $S(\hat{\rho}) = -\mathrm{Tr}[\hat{\rho}\log\hat{\rho}]$. Substituting equations (46) and (47) into this equation and calculating the von Neumann entropies, we find that the Holevo capacity C_{H} is given by

$$\begin{aligned} C_{\mathrm{H}} = &-\tfrac{1}{2}\left[1+\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}\right]\log\left[1+\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}\right] \\ &-\tfrac{1}{2}\left[1-\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}\right]\log\left[1-\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}\right] + \tfrac{1}{2}\left(1-\gamma_t^2\right)\log\left(1-\gamma_t^2\right) \\ &+\tfrac{1}{4}\left(1-\gamma_t^2\right)(1+\langle\hat{\sigma}_z\rangle_{\mathrm{eq}})\log(1+\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}) \\ &+\tfrac{1}{4}\left(1-\gamma_t^2\right)(1-\langle\hat{\sigma}_z\rangle_{\mathrm{eq}})\log(1-\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}) \\ &+\tfrac{1}{4}\left[1+\gamma_t^2+\sqrt{(2\gamma_t)^2+\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2}\right] \\ &\times\log\left[1+\gamma_t^2+\sqrt{(2\gamma_t)^2+\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2}\right] \\ &+\tfrac{1}{4}\left[1+\gamma_t^2-\sqrt{(2\gamma_t)^2+\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2}\right] \\ &\times\log\left[1+\gamma_t^2-\sqrt{(2\gamma_t)^2+\left(1-\gamma_t^2\right)\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}^2}\right]. \end{aligned} \tag{71}$$

In particular, when $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = 0$, the Holevo capacity is simplified as

$$C_{\mathrm{H}|\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}=0} = (1+\gamma_t)\log(1+\gamma_t) + (1-\gamma_t)\log(1-\gamma_t). \tag{72}$$

The Holevo capacity C_{H} is plotted as the function of the transmission time t of the encoded qubit in figure 5. The figure shows that the non-Markovian effect suppresses the decay of the classical information capacity of the quantum dense coding system.

When Bob extracts the classical information encoded by Alice from the received state by means of the Bell-state measurement, the channel matrix of the quantum dense coding system is given by

$$\mathsf{P}_t = \frac{1}{4}\begin{pmatrix} (1+\gamma_t)^2 & (1-\gamma_t)^2 & 1-\gamma_t^2 & 1-\gamma_t^2 \\ (1-\gamma_t)^2 & (1+\gamma_t)^2 & 1-\gamma_t^2 & 1-\gamma_t^2 \\ 1-\gamma_t^2 & 1-\gamma_t^2 & (1+\gamma_t)^2 & (1-\gamma_t)^2 \\ 1-\gamma_t^2 & 1-\gamma_t^2 & (1-\gamma_t)^2 & (1+\gamma_t)^2 \end{pmatrix} \tag{73}$$

which provides the Shannon mutual information

$$I_{\mathrm{Bell}} = (1+\gamma_t)\log(1+\gamma_t) + (1-\gamma_t)\log(1-\gamma_t). \tag{74}$$

Then we find from equation (72) that the following equality holds

$$C_{\mathrm{H}|\langle\hat{\sigma}_z\rangle_{\mathrm{eq}}=0} = I_{\mathrm{Bell}}. \tag{75}$$

This result means that the quantum block-coding and collective decoding [34–36] do not work for the super-additivity of the mutual information when the temperature of the thermal reservoir is infinite ($T \to \infty$ or equivalently $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = 0$).

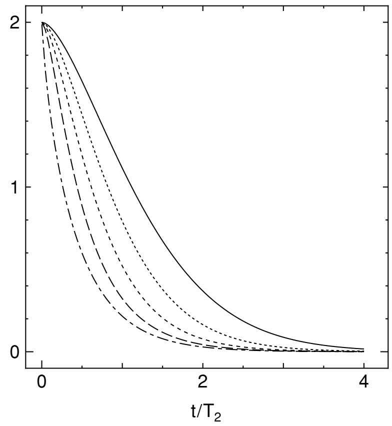

Figure 5. The Holevo capacity C_{H} of the quantum dense coding system, where $\langle\hat{\sigma}_z\rangle_{\mathrm{eq}} = -1/2$ and the information is measured in bits. In the figure, the solid line represents $\tau_R/T_2 = 2.0$, the dotted line $\tau_R/T_2 = 1.0$, the short dashed line $\tau_R/T_2 = 0.5$, the dashed line $\tau_R/T_2 = 0.2$, and the dot-dashed line $\tau_R/T_2 = 0.0$ which corresponds to the Markovian approximation.

7. Concluding remarks

In this paper, we have investigated the decoherence of quantum information of qubits under the influence of the non-Markovian quantum channel which is derived from the system–reservoir model by eliminating reservoir variables by means of the projector operator method in the time-convolutionless formalism, where the correlation time of the reservoir variables is assumed to take a finite value. We have evaluated the degradation of purity, distinguishability and entanglement of qubit states and we have found that the non-Markovian effect suppresses their degradation. Furthermore we have investigated the quantum teleportation and the quantum dense coding of qubits under the influence of the non-Markovian quantum channel. To show how faithfully a qubit state is teleported, we have calculated the average fidelity between the teleported state and the original state. We have found that the entanglement shared by the sender and receiver makes the fidelity greater than that obtained for the classical teleportation and the non-Markovian effect improves the performance of the quantum teleportation. In the quantum dense coding, we have calculated the Holevo capacity and compared it with the Shannon mutual information obtained in the case that the receiver performs the Bell-state measurement to extract the classical information encoded by the sender. The non-Markovian effect also improves the Holevo capacity. We have found that the Holevo capacity becomes equal to the Shannon mutual information when the temperature of the thermal reservoir is infinite. In this paper, we have confined ourselves to focusing our attention on qubit systems. Continuous variable quantum information under the influence of the non-Markovian quantum channel will be considered elsewhere.

Appendix. Correlation functions of reservoir variables

This appendix derives equations (12) and (13) by making use of the microscopic model of the thermal reservoir. When the thermal reservoir is a set of bosonic oscillators, the reservoir variable $\hat{R}$ appeared in equations (4) and (3) is given by $\lambda\hat{R} = \sum_k \lambda_k \hat{a}_k$, where $\hat{a}_k$ is the bosonic annihilation operator of the kth mode. Since the thermal reservoir is in the thermal equilibrium state, we obtain the functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$

$$\phi_{+-}(t) = \sum_k \lambda_k^2 \int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega_Q \tau} \langle \hat{a}_k^\dagger(\tau)\hat{a}_k(0)\rangle_{\mathrm{eq}} \tag{A.1}$$

$$\phi_{-+}(t) = \sum_k \lambda_k^2 \int_0^t \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega_Q \tau} \langle \hat{a}_k(\tau)\hat{a}_k^\dagger(0)\rangle_{\mathrm{eq}}. \tag{A.2}$$

We assume here that the time-evolution of the bosonic oscillators of the thermal reservoir is subject to the quantum master equation. In other word, they are assumed to be the damped oscillators. In this case, when the initial state of the damped oscillators is the thermal equilibrium state with temperature T, we can obtain the correlation functions

$$\langle \hat{a}_k(t)\hat{a}_k^\dagger(0)\rangle_{\mathrm{eq}} = (\bar{n}_k + 1)\, \mathrm{e}^{-\mathrm{i}\omega_k t - t/\tau_k} \tag{A.3}$$

$$\langle \hat{a}_k^\dagger(t)\hat{a}_k(0)\rangle_{\mathrm{eq}} = \bar{n}_k\, \mathrm{e}^{-\mathrm{i}\omega_k t - t/\tau_k} \tag{A.4}$$

where $\bar{n}_k = (\mathrm{e}^{\hbar\omega_k/k_{\mathrm{B}}T} - 1)^{-1}$. In these equations, ω_k and τ_k are the frequency and decay time of the k-mode of the damped oscillator. Substituting equations (A.3) and (A.4) into equations (A.1) and (A.2), we obtain the functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$

$$\phi_{+-}(t) = \sum_k \lambda_k^2 \bar{n}_k \tau_k \frac{1 - \exp(-(1 + \mathrm{i}(\omega_Q - \omega_k)\tau_k)t/\tau_k)}{1 + \mathrm{i}(\omega_Q - \omega_k)\tau_k} \tag{A.5}$$

$$\phi_{-+}(t) = \sum_k \lambda_k^2 (\bar{n}_k + 1)\tau_k \frac{1 - \exp(-(1 - \mathrm{i}(\omega_Q - \omega_k)\tau_k)t/\tau_k)}{1 - \mathrm{i}(\omega_Q - \omega_k)\tau_k}. \tag{A.6}$$

Let us introduce the spectral density $D(\omega)$ of the thermal reservoir by the relation

$$D(\omega) = \frac{1}{\pi} \sum_k \lambda_k^2 \tau_k \delta(\omega - \omega_k). \tag{A.7}$$

Then the functions $\phi_{+-}(t)$ and $\phi_{-+}(t)$ can be expressed as

$$\phi_{+-}(t) = \pi \int \mathrm{d}\omega D(\omega)\bar{n}(\omega) \frac{1 - \exp(-(1 + \mathrm{i}(\omega_Q - \omega)\tau(\omega))t/\tau(\omega))}{1 + \mathrm{i}(\omega_Q - \omega)\tau(\omega)} \tag{A.8}$$

$$\phi_{-+}(t) = \pi \int \mathrm{d}\omega D(\omega)[\bar{n}(\omega) + 1] \frac{1 - \exp(-(1 - \mathrm{i}(\omega_Q - \omega)\tau(\omega))t/\tau(\omega))}{1 - \mathrm{i}(\omega_Q - \omega)\tau(\omega)}. \tag{A.9}$$

If the spectral density $D(\omega)$ has a very sharp peak around the transition frequency ω_Q of the qubit, that is, $D(\omega) \approx D\delta(\omega - \omega_Q)$, we obtain

$$\phi_{+-}(t) = \pi D\bar{n}(\omega_Q)(1 - \mathrm{e}^{-t/\tau_R}) \tag{A.10}$$

$$\phi_{-+}(t) = \pi D[\bar{n}(\omega_Q) + 1](1 - \mathrm{e}^{-t/\tau_R}) \tag{A.11}$$

which are equal to equations (12) and (13). In these equations, we put $\tau_R = \tau(\omega_Q)$. Another model of the thermal reservoir that yields equations (12) and (13) has been considered in [14].

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